Exercise 5I: Solutions

1 a $\{1,3,4\}$

b $\{1,3,4,5,6\}$

c $\{4\}$

d $\{1,2,3,4,5,6\}$

e 3

f $\varnothing$, $\{4\}$, $\{5\}$, $\{6\}$, $\{4,5\}$, $\{4,6\}$, $\{5,6\}$, $\{4,5,6\}$

2 Let T be the set of track athletes and F be the set of field athletes. Then the number of athletes in the team will be given by:

$$\begin{aligned}|T \cup F| &= |T| + |F| - |T \cap F| \\ &= 25 + 23 - 12 \\ &= 36.\end{aligned}$$

3 Let A be the set of patients taking medication A and let B be the set of patients taking medication B. Then,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ 50 &= 25 + 29 - |A \cap B| \\ 50 &= 54 - |P \cap C| \\ \therefore |A \cap B| &= 4.\end{aligned}$$

4 Let A and B be the sets comprising of multiples 7 and 9 respectively. Clearly $A \cap B$ consists of the multiples of 7 and 9, that is, multiples of 63. Therefore, $|A| = 90, |B| = 70$ and $|A \cap B| = 10$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= 90 + 70 - 10 \\ &= 150.\end{aligned}$$

5 a Let A and B be the sets comprising of multiples 2 and 3 respectively. Clearly $A \cap B$ consists of the multiples of 2 and 3, that is, multiples of 6. Therefore, $|A| = 48, |B| = 32$ and $|A \cap B| = 16$. We then use the Inclusion Exclusion Principle to find that,

$$\begin{aligned}|A \cup B| &= |A| + |B| - |A \cap B| \\ &= 48 + 32 - 16 \\ &= 64.\end{aligned}$$

b We have already found that 64 are divisible by 2 or 3. Therefore, $96 - 64 = 32$ will not be divisible by 2 or 3.

6 a There are three vowels so there are 3 choices for the first letter, 4 for the second letter, 3 for the third, and so on. This gives $3 \times 4 \times 3 \times 2 \times 1 = 72$ arrangements.

b There are three vowels so there are 3 choices for the last letter, 4 for the first letter, 3 for the second, and so on. This gives $4 \times 3 \times 2 \times 1 \times 3 = 72$ arrangements.

c There are three vowels so there are 3 choices for the first letter and then 2 for the last letter. As two letters have be used, there are then 3 choices for the second letter, 2 for the third, and 1 for the fourth. This gives $3 \times 3 \times 2 \times 1 \times 2 = 18$ arrangements.

d We let A be the set of arrangements that begin with a vowel, and B be the set of arrangements that end with a vowel. Then $A \cap B$ is the set of arrangements that begin and end with a vowel. We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 72 + 72 - 36 \\ &= 108. \end{aligned}$$

7 a There are ten perfect squares,
$\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$.

There are 4 perfect cubes,

$\{1, 8, 27, 64\}$.

Only 1 and 64 are common to both lists, so there are $10 + 4 - 2 = 12$ integers that are perfect squares or perfect cubes.

b Let A be the set of perfect squares. Since $31^2 = 961$ and $32^2 = 1024$, there are exactly 31 perfect squares no greater than 1000,

$$A = \{1^2, 2^2, \ldots, 31^2\}.$$

Let B be the set of perfect cubes. Since $10^3 = 1000$, there are exactly 10 perfect cubes no greater than 1000,

$$B = \{1^3, 2^3, \ldots, 10^3\}.$$

Clearly $A \cap B$ consists of integers that are perfect powers of 2 and 3, that is, powers of 6. Since $3^6 = 729$ and $4^6 = 4096$, there are exactly 3 powers of 6 no greater than 1000,

$A \cap B = \{1^6, 2^6, 3^6\}$.

We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 31 + 10 - 3 \\ &= 38. \end{aligned}$$

8 Let sets A, B and C consist of those integers that are divisible by $2, 3$ and 5 respectively. We then have

set	multiples of	size
A	2	$\|A\| = 60$
B	3	$\|B\| = 40$
C	5	$\|C\| = 24$
$A \cap B$	6	$\|A \cap B\| = 20$
$A \cap C$	10	$A \cap C\| = 12$
$B \cap C$	15	$\|B \cap C\| = 8$
$A \cap B \cap C$	30	$\|A \cap B \cap C\| = 4$

We then use the Inclusion-Exclusion Principle to give,

$$\begin{aligned} &|A \cup B \cup C| \\ =&|A| + |B| + |C| \\ -&|A \cap B| - |A \cap C| - |B \cap C| \\ +&|A \cap B \cap C| \\ =&60 + 40 + 24 - 20 - 12 - 8 + 4 \\ =&88 \end{aligned}$$

Therefore, there are 88 integers from 1 to 120 inclusive that are divisible by $2, 3$ or 5.

9 Let sets A, B and C consist of those integers that are divisible by of $2, 5$ and 11 respectively. We then have

set	multiples of	size of set
A	2	$\lvert A\rvert = 110$
B	5	$\lvert B\rvert = 44$
C	11	$\lvert C\rvert = 20$
$A \cap B$	10	$\lvert A \cap B\rvert = 22$
$A \cap C$	22	$A \cap C\rvert = 10$
$B \cap C$	55	$\lvert B \cap C\rvert = 4$
$A \cap B \cap C$	110	$\lvert A \cap B \cap C\rvert = 2$

We then use the Inclusion-Exclusion Principle to give,

$$\begin{aligned} &|A \cup B \cup C| \\ =&|A| + |B| + |C| \\ -&|A \cap B| - |A \cap C| - |B \cap C| \\ +&|A \cap B \cap C| \\ =&110 + 44 + 20 - 22 - 10 - 4 + 2 \\ =&140 \end{aligned}$$

Therefore, there are 140 integers from 1 to 120 inclusive that are divisible by $2, 3$ or 5, and $220 - 140 = 80$ integers that are not.

10 Let B be the set of students that study biology, let P be the set of students that study physics, and let C be the set of students that study chemistry. Then,

$$\begin{aligned} &|B \cup P \cup C| \\ &= |B| + |P| + |C| \\ &- |B \cap P| - |B \cap C| - |P \cap C| \\ &+ |B \cap P \cap C| \\ 90 =&36 + 42 + 40 - 9 - 8 - 7 + |B \cap P \cap C| \\ 90 =&94 - |B \cap P \cap C| \end{aligned}$$

Therefore, $|B \cap P \cap C| = 4$.

11a There are ${}^4C_2$ ways of choosing 2 Year Ten students from 4, and ${}^9C_4$ ways of choosing 4 more students from those in Years Eleven and Twelve. This gives a total of ${}^4C_2 \times {}^9C_4 = 756$ selections.

b There are ${}^5C_2$ ways of choosing 2 Year Eleven students from 5, and ${}^8C_4$ ways of choosing 4 more students from those in Years Ten and Twelve. This gives a total of ${}^5C_2 \times {}^8C_4 = 700$ selections.

c There are ${}^4C_2$ ways of choosing 2 Year Ten students from 4. There are ${}^5C_2$ ways of choosing 2 from Year Eleven. There are ${}^4C_2$ ways of choosing 2 remaining students from Year Twelve. This gives a total of ${}^4C_2 \times {}^5C_2 \times {}^4C_2 = 360$ selections.

d We let A be the set of selections with exactly 2 Year Ten students and let B be the set of selections with exactly 2 Year Eleven students. Then $A \cap B$ is set selection with exactly 2 Year Ten and Year Eleven students. We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 756 + 700 - 360 \\ &= 1096. \end{aligned}$$

12 We let A be the set of hands with exactly one heart. Then there are ${}^{13}C_1$ ways of choosing the heart and ${}^{39}C_4$ ways of choosing the 4 non-heart cards. Therefore,

$$|A| = {}^{13}C_1 \times {}^{39}C_4 = 1069263$$

We let B be the set of hands with exactly two diamonds. Then there are ${}^{13}C_2$ ways of choosing the two diamonds and ${}^{39}C_3$ ways of choosing the 3 non-diamond cards. Therefore,

$$|B| = {}^{13}C_2 \times {}^{39}C_3 = 712842$$

Then $A \cap B$ is the set of hands with exactly one heart and two diamonds. Then there are ${}^{13}C_1$ ways of choosing the one heart, ${}^{13}C_2$ ways of choosing the two diamonds and ${}^{26}C_2$ ways of choosing the 2 remaining cards. Therefore,

$$|A \cap B| = {}^{13}C_1 \times {}^{13}C_2 \times {}^{26}C_2 = 329550.$$

We then use the Inclusion-Exclusion Principle to find that,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 1069263 + 712842 - 329550 \\ &= 1452555. \end{aligned}$$

13 The sum of numbers divisible by 2 will be

$$2 + 4 + \ldots + 100 = 2550.$$

The sum of numbers divisible by 3 will be

$$3 + 6 + \ldots + 99 = 1683.$$

Adding these together does not give the desired answer, since it double counts those numbers divisible by 2 and 3, that is, those divisible by 6. The sum of these numbers is

$$6 + 12 + \ldots + 96 = 816.$$

Therefore, the required answer is $2550 + 1683 - 816 = 3417$.

14 Let F, C and G be the sets of students studying French, Chinese and German, respectively. Then using the Exclusion-Inclusion Principle gives,

$$\begin{aligned} & |F \cup C \cup G| \\ =& |F| + |C| + |G| \\ & - |F \cap C| - |F \cap G| - |C \cap G| \\ & + |F \cap C \cap G| \\ 80 =& 30 + 45 + 30 - |F \cap C| - |F \cap G| - |C \cap G| + 15 \\ 80 =& 120 - |F \cap C| - |F \cap G| - |C \cap G| \\ 40 =& |F \cap C| + |F \cap G| + |C \cap G| \end{aligned}$$

Now $|F \cap C| + |F \cap G| + |C \cap G|$ counts the number of students who study at two subjects, but triple counts those who study three subjects. Therefore, the number that study exactly 2 subjects will be $40 - 2 \times 15 = 10$.